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# Walks in Rigid Environments: Continuous Limits

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We derive the continuous limits of kinetic equations for spatially discrete systems generated by the motion of a particle in a random array of scatterers. The type of scatterer at a vertex changes after the $r$-th visit of the particle to this vertex, where $1 \leqslant r \leqslant \infty$. Such deterministic cellular automata belong to the class of walks in rigid environments. It has been recently shown that they form the simplest dynamical models with sub-diffusive, diffusive and super-diffusive behaviour. Due to the deterministic character of the dynamics, the continuous limit equations obtained for these models are of the Euler type rather than the diffusive type. The reason for that is that the fluctuations in these models are relatively small and there is no scaling of probabilities similar, for example, to those in the case of biased random walk, that can account for them.

**KEY WORDS:** Lorentz lattice gas; first visit equation; oscillation; propagation; rigid environment.

## 1. INTRODUCTION

Walks in rigid environments[1,2] belong to the class of *deterministic walks in random environments* which includes systems generated by the deterministic motion of a particle in an environment of scatterers randomly distributed over the vertices of some graph $G$. The dynamics of such systems allows for the particle's feedback on the environment. In other words, the passage of a particle can modify the type of scatterer located at the visited vertex, according to some deterministic rule.

The basic characteristic of a rigid environment is the function $r: G \to \mathbb{Z}_+$. The value $r(g)$ of this function at a given vertex $g \in G$ is called

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the local rigidity of the environment at $g$. This means that the scatterer located at $g$ changes its type after the $r(g)$-th visit of the particle to that vertex. In the case when $r(g) = \infty$ for all $g \in G$, the scatterers do not change their type. This case corresponds to the deterministic Lorentz lattice gas.[3] Walks in rigid environments have been extensively studied in various fields including computer networks, evolutionary dynamics etc., and they served as a paradigm for signal propagation in random media.[4–7]

In this paper we consider systems with one moving particle. It has been shown[1, 2] that one particle walks in rigid environments are completely solvable in the case when $G$ is a one-dimensional lattice $\mathbb{Z}$ and the rigidity of the environment is constant: $r(z) = r$ for all $z \in \mathbb{Z}$. There are four types of scatterers on $\mathbb{Z}$: forward- and back-scatterers which respect the reflection symmetry of $\mathbb{Z}$, and right- and left-rotators which do not. It has also been demonstrated in refs. 1 and 2 that walks in rigid environments, formed by either the first or the second pair of scatterers on $\mathbb{Z}$, can exhibit sub-diffusive, diffusive or super-diffusive behaviour. (The last case corresponds to the ultimate propagation on $\mathbb{Z}$ with random velocity.[8, 1]) In fact, these systems can be viewed as the simplest, completely solvable, dynamical models of sub-/super-/diffusive motion.

In this paper we study continuous limits of these models and derive the corresponding kinetic equations. The results show that in the limit the deterministic dynamics prevails over random fluctuations caused by the randomness in the initial distribution of scatterers. This should be contrasted, for example, with the continuous limit of the biased random walk on $\mathbb{Z}$ where the diffusive behaviour emerges on top of the propagation via a proper scaling of transition probabilities (see, e.g., ref. 9). In the walks in rigid environments examined here, no such scaling exists. The reason for that is the character of evolution in these models. At any moment of time, the environment consists of two parts: the region already visited by the particle and its complement. The deterministic motion in the visited region is intermittent with random excursions to its complement. By virtue of such excursions the deterministic (already visited) region is growing and, consequently, periods of deterministic motion grow as well, which makes random fluctuations relatively small.

## 2. WALKS IN RIGID ENVIRONMENTS

We now give a formal definition of a walk in a rigid environment.[1, 2] Let $G$ be a simple regular undirected infinite graph. For a given vertex $g$, a scatterer is defined as a map $s_g$: $A_{\text{in}}(g) \to A_{\text{out}}(g)$ from the set $A_{\text{in}}(g)$ of all incoming edges of $g$ to the set $A_{\text{out}}(g)$ of all edges originating at $g$. Since $G$ is undirected, the sets $A_{\text{in}}(g)$ and $A_{\text{out}}(g)$ can be identified.

We denote as $S$ the set of all possible scatterers on $G$. It is easy to see that card $S = d^d$ where $d$ is the degree of the regular graph $G$.

A walk in a rigid environment is a dynamical model that is described by the following characteristics:

– a graph $G$;

– a subset $\hat{S} \subseteq S$ of the scatterers allowed in the model;

– a function $r: G \to \mathbb{Z}_+$ called the rigidity of the environment;

– a collection of functions $\{e_g\}_{g \in G}$ where each $e_g: \hat{S} \to \hat{S}$ defines the "flipping rule" according to which the scatterer at $g$ changes its type (flips). In other words, it defines the order in which scatterers of different types appear at $g$.

To say that the environment has constant rigidity $r$ means that the scatterer at vertex $g \in G$ changes its type (flips) after $r$ visits of the particle to this vertex. We define the index $i$ of the scatterer at $g$ as the number of visits of the particle to $g$ that occured since the last time when this scatterer flipped. Hence, the state of the scatterer at vertex $g$ at any time $t$ is determined by two quantities: the type of the scatterer and its index. We will define the state function $\eta: G \times \mathbb{Z} \to \Phi_r$ as:

$$\eta(g, t) = (s, i),$$

where $(s, i) \in \Phi_r$, and $\Phi_r = \hat{S} \times \mathbb{Z}_r$ is the set of possible states of a scatterer (the second factor in the product denotes the cyclic group of order $r$ with mod $r$ addition). After each visit of the particle to $g$ the state of the scatterer changes according to the rule given by a function $\phi: \Phi_r \to \Phi_r$ which is defined as:

$$\phi(s, i) = \begin{cases} (s, i+1), & \text{if } \ 0 \leqslant i < r-1, \\ (e(s), 0), & \text{if } \ i = r-1. \end{cases}$$

In the next section we will introduce two models of walks in rigid environments on $\mathbb{Z}$.

## 3. FINITE-DIFFERENCE EQUATIONS

We consider the motion of a single particle on the real line $\mathbb{R}$ where each integer point $z \in \mathbb{R}$ is occupied by a scatterer of some type. The scatterers are distributed independently over all such points. A particle is moving with the unit speed on $\mathbb{R}$, i.e., $|v(t)| = 1$ at any moment of time $t$. The velocity vector $v(t)$ of the particle may change upon a collision with a scatterer. By $z(t)$ we denote the position of the particle at time $t$. It makes sense to consider a discrete version of this model. We assume that the particle is moving on a lattice, which can be identified with $\mathbb{Z}$. At each integer moment of time the particle hops from the current vertex $z(t) \in \mathbb{Z}$ to one of its neighboring vertices changing the state of the scatterer at $z(t)$. The choice of neighbor is completely determined by the velocity vector of the particle $v(t)$ and the type of scatterer at position $z(t)$. To distinguish between the two moments of time just before and just after the particle's interaction with the scatterer we denote them by $t$ and $t_+$ respectively. Formally, the dynamics of walks in rigid environments on $\mathbb{Z}$ can be described by the following equations:

$$z(t+1) = z(t) + v(t),$$

$$v(t+1) = w_{s(t)}(v(t)),$$

$$\eta(z, t+1) = \begin{cases} \eta(z, t) & \text{if} \quad z \neq z(t), \\ \phi(\eta(z(t), t)) & \text{if} \quad z = z(t). \end{cases}$$

Here, the function $w_{s(t)}$ is completely defined by the type of scatterer $s(t)$ at position $z(t)$ and can be written explicitly for each model.

The set $S$ of all possible scatterers on $\mathbb{Z}$ consists of the following four types: forward- and back-scatterers, and right and left rotators. The forward scatterer corresponds to the trivial case when the velocity vector of the particle does not change upon a collision, i.e., $v(t_+) = v(t)$. The back-scatterer changes the velocity vector of the particle to the opposite, i.e., $v(t_+) = -v(t)$. The right and left rotators send the scattered particle to the right or to the left respectively, i.e., $v(t_+) = 1$ for the right rotator and $v(t_+) = -1$ for the left rotator. The first pair of scatterers will be referred to as non-oriented scatterers while the second pair as oriented scatterers. In this paper we will examine both the model with oriented scatterers (*OS-model*) and the model with non-oriented scatterers (*NOS-model*). In either case the environment is assumed to have constant rigidity $r \geqslant 1$.[1]

We will derive probabilistic equations governing the dynamics of the particle in OS- and NOS-models and find the continuous limits of those equations. Given that the inverse dynamics is generally not defined and, in this case only the NOS-model has well-defined dynamics for all $t \in \mathbb{Z}$, we will only consider $t \geqslant 0$. We will assume that the particle starts at the origin with $v(0) = v(0_+) = 1$, and all scatterers on $\mathbb{Z}$ have index 0 at time $t = 0$. We denote as $f(z, t)$ the probability of finding the particle at position $z$ for the first time at time $t$. Then we can write down the formula of total probability:

$$f(z+1, t+1) = \sum_s a_s f(z, t - t_s) \tag{1}$$

where $a_s$ is the probability that the particle propagates from $z$ to $z+1$ in $t_s+1$ time steps, $\sum_s a_s = 1$, and the summation is performed over all possible time delays $t_s$ (i.e., all possible loops that the particle can make between two successive visits to position $z$). A similar equation can be written for the propagation along the negative axis.

Let us take a closer look at the conditions under which Eq. (1) makes sense. These conditions reflect the hybrid nature of walks in random environments which are neither purely stochastic nor purely deterministic systems.

At any time instant $t > 0$ the lattice $\mathbb{Z}$ can be divided into two subsets: a subset $D_t$ formed by all sites $z \in \mathbb{Z}$ visited by the particle to the moment $t$ and its complement $\bar{D}_t$. Note that the dynamics of the particle in $D_t$ is purely deterministic. To reflect this fact we will call $D_t$ a *deterministic region*. Randomness is introduced into the dynamics when the particle visits sites in $\bar{D}_t$. Indeed, since the initial distribution of scatterers on $\mathbb{Z}$ is random then the first visit of the particle to any site can be interpreted as a random event. We will refer to $\bar{D}_t$ as a *random region*. Initially the entire lattice forms a random region: $\bar{D}_0 = \mathbb{Z}$. But once the particle starts moving, a non-trivial deterministic region appears. Generally, $D_0 \subset D_1 \subseteq D_2 \subseteq \cdots \subseteq D_n \subseteq \cdots$. We denote the union of all these deterministic regions as $D_\infty = \bigcup_t D_t$. Then, for Eq. (1) to make sense, the following condition has to be satisfied:

**Condition I (Unboundedness).** $D_\infty$ is unbounded, i.e., the particle never performs periodic motion.

If this condition is not met, then at some time $t^*$ the particle starts moving deterministically, where $t^*$ depends on the initial configuration of scatterers. Hence, it does not make sense to speak about the probability of finding the particle at any location for $t > t^*$. For the two models under study, however, the unboundedness of $D_\infty$ has been proven for all $r < \infty$.[1, 2]

In general, equations similar to (1) can be written for propagation along any unbounded path on any graph satisfying the assumptions made at the beginning of Section 2, provided that the particle visits all vertices on this path. These equations are rather general: (1) is nothing but the formula of total probability. Equations of this type have been derived for many probabilistic models. For walks in rigid environments, however, only the probability of the first visit of the particle to a site makes sense. Indeed, if the first visit of the particle to position $z$ occured at some $t_z$, then for all $t > t_z$, $z$ will belong to the deterministic region, and so, we can no longer speak about the probability of finding the particle at $z$.

Equation (1) first appeared in ref. 10 with an additional restriction that the sum in the r.h.s. of the equation be over a finite number of possible time delays. We will drop this restriction and consider a more general class of cellular automata where the summation can be performed over an infinite number of configurations.

In more general case, when the distance between neighboring sites on $\mathbb{R}$ is $\zeta$, and $\tau$ is the elementary time step, the equation for $f(z, t)$ reads:[10]

$$f(z+\zeta, t+\tau) = \sum_s a_s f(z, t-t_s\tau). \tag{2}$$

Here $a_s$ is the probability that the particle propagates from $z$ to $z+\zeta$ in $t_s+1$ time steps. Again the sum of these probabilities over all possible time delays is 1.

These equations will serve as a basis for deriving specific equations for each of the above models.

### 3.1. NOS Model. Odd Rigidity

To derive the equation governing the dynamics of the particle in the NOS-model we need to understand what a typical trajectory of the particle looks like. We define the state of the scatterer at position $z$ at time $t$ as $\eta(z, t) = (s, i)$ where $s \in \{0, 1\}$ (0 corresponds to forward-scatterer and 1 to back-scatterer) and $0 \leqslant i < r$.

The finite-difference equation for $r = 1$ has been written in ref. 8. We will derive it here for the sake of completeness. Note that in this case the notion of the index of a scatterer is redundant because the scatterer flips every time it is hit by the particle. Thus, the state of the scatterer is determined only by its type, i.e., $\eta(z, t) = s$. Let us suppose now that the particle arrives at position $z$ at time $t$. If $\eta(z, t) = 0$ then the particle will continue moving with the same velocity $v(t)$ and arrive at $z+1$ at time $t+1$. The state of the scatterer at $z$ will change to to $\eta(z, t_+) = 1$. If $\eta(z, t) = 1$ then, at first, the particle will be reflected to position $z-1$ and the scatterer at $z$ will flip. Upon arriving at $z-1$, however, the particle will be reflected back to position $z$. At this point the scatterer at $z$ is in state $\eta(z, t+2) = 0$. Hence, the particle will continue moving in the positive direction and arrive at $z+1$ at time $t+3$.

Let $q$ denote the probability that a site is occupied by a forward scatterer at $t = 0$ and $p = 1-q$. Then the two possible time delays $t_s$ and their probabilities $a_s$ in Eq. (1) are given by:

$$\begin{aligned} t_1 &= 0, \qquad a_1 = q, \\ t_2 &= 2, \qquad a_2 = p, \end{aligned} \tag{3}$$

and the first visit equation reads:

$$f(z+1, t+1) = qf(z, t) + pf(z, t-2).$$

Next, we will consider the case when $r > 1$. Let us suppose that at time $t$ the particle arrives at position $z$ for the first time. Again, there are two possible outcomes.

If $\eta(z, t) = (0, 0)$, then after the particle's interaction with the scatterer the state of that scatterer will change to $\eta(z, t_+) = (0, 1)$. The particle will continue moving with velocity $v(t)$ and arrive at position $z+1$ at time $t+1$.

If $\eta(z, t) = (1, 0)$, then the particle will be reflected by the scatterer at position $z$ and start traveling in the negative direction until it collides with another back-scatterer, say at position $y < z$. Upon reflection by this back-scatterer the particle will bounce between $z$ and $y$ until all scatterers on the interval $(y, z)$ flip and assume state $(1, 0)$. Thus, the distance between positions $z$ and $y$ will be covered by the particle $r-1$ times. Meanwhile the state of the scatterer at position $z$ changes to $(1, (r+1)/2)$. To flip this scatterer and move to the next position on the lattice the particle has to make additional $(r+1)/2$ visits to $z-1$. Let $T_{z,z+1}$ denote the time that it takes the particle to propagate from $z$ to $z+1$. Then the configuration of scatterers at time $t+T_{z,z+1}$ is given by:

$$\begin{aligned} \eta(x, t+T_{z,z+1}) &= (1, 0), \qquad y < x < z-1, \\ \eta(z-1, t+T_{z,z+1}) &= \left(1, \frac{r+1}{2}\right), \\ \eta(z, t+T_{z,z+1}) &= (0, 1). \end{aligned}$$

**Remark.** Note that between times $t$ and $t+T_{z,z+1}$ the scatterer at position $y$ is visited $(r-1)/2$ times. Hence, this is also the number of times that position $z-1$ will be revisited by the particle if, in the course of propagation, it encounters another back-scatterer.

There is one additional case that needs to be considered: when at, $t=0$, the scatterers at positions 0 and 1 are in the state (1, 0). In this case, upon arriving at $z=1$ for the first time at $t=1$ the particle will simply bounce between 0 and 1 until *both* scatterers flip. It will arrive at position 2 after $T_{1,2} = 2r+1$ time steps. The configuration of scatterers resulting from the propagation of the particle from position 1 to position 2 is given by:

$$\begin{aligned} \eta(0, 1+T_{1,2}) &= (0, 0), \\ \eta(1, 1+T_{1,2}) &= (0, 1). \end{aligned}$$

**Remark.** Note that in the course of propagation, the scatterer at position 0 is visited $r$ times which does not follow the rule described in the previous Remark. However this is the only exception to that rule.

We are ready now to compute the time delays $t_s$ in Eq. (1) and the probabilities $a_s$ with which they occur. We start with $\eta(z,t)=(1,0)$. Let $d$ be the random variable that corresponds to the distance between $z$ and the position $y$ of the back-scatterer closest to $z$ with $y<z$, at time $t$. Note that for all $z>1$ this distance is bounded from below by 2.

It follows from the above argument that the time required for the particle to propagate from $z$ to $z+1$, for each $d=j$, is equal to

$$t_j=(r-1)\,j+(r+1), \qquad j\geqslant 2. \tag{4}$$

The corresponding probabilities $a_j=\mathscr{P}\{d=j\}$ can be computed as follows:

$$\begin{aligned} a_j &= \mathscr{P}\{\eta(z,t)=(1,*);\ \eta(z-j,t)=(1,*);\\ &\qquad \eta(n,t)=(0,*),\ z-j<n<z\}, \qquad j\geqslant 2\\ &= \begin{cases} \mathscr{P}\{\eta(z,0)=\eta(z-j+1,0)=(1,0);\\ \quad \eta(n,0)=(0,0),\ z-j+1<n<z\}, \qquad 2\leqslant j\leqslant z-1;\\ \mathscr{P}\{\eta(z,0)=\eta(1,0)=(1,0);\ \eta(n,0)=\eta(0,0)=(0,0),\ 1<n<z\}\\ \quad +\mathscr{P}\{\eta(z,0)=\eta(0,0)=(1,0);\ \eta(n,0)=(0,0),\ 0<n<z\}, \qquad j=z;\\ \mathscr{P}\{\eta(z,0)=\eta(z-j,0)=(1,0);\ \eta(n,0)=(0,0),\ z-j<n<z\}\\ \quad +\mathscr{P}\{\eta(z,0)=\eta(z-j,0)=\eta(0,0)=\eta(1,0)=(1,0);\\ \quad\ \eta(n,0)=(0,0),\ z-j<n<z,\ n\neq 0,1\}, \qquad j\geqslant z+1. \end{cases} \end{aligned} \tag{5}$$

where $*$ denotes *any* index between 0 and $r-1$.

If, on the other hand, position $z$ is occupied by a forward scatterer at time $t$, i.e., $\eta(z,t)=(0,0)$, then there is no time delay in propagation from $z$ to $z+1$. To conform to the notations used in Eq. (1) we will assign subscript $j=1$ to the quantities corresponding to this case. Thus,

$$t_1=0 \tag{6}$$

and the corresponding probability is given by

$$a_1=\mathscr{P}\{\eta(z,t)=(0,*)\}.$$

Let us recall that $q$ is the probability that a site is occupied by a forward scatterer at time $t=0$ and $p=1-q$. Then Eq. (5) allows us to compute probabilities $a_j$ in terms of these two quantities:

$$a_j = \begin{cases} q, & j=1; \\ p^2 q^{j-2}, & 2 \leqslant j \leqslant z-1; \\ p^2 q^{j-1} + p^2 q^{j-1} = 2p^2 q^{j-1}, & j=z; \\ p^2 q^{j-1} + p^4 q^{j-3} = p^2(p^2+q^2)\, q^{j-3}, & j \geqslant z+1. \end{cases} \tag{7}$$

It is easy to verify that $\sum_j a_j = 1$.

Now, combining Eqs. (1), (6), and (7) we can write out the first-visit equation for the NOS-model with odd rigidity:

$$f(z+1, t+1) = q f(z, t) + \sum_{j=2}^{\infty} a_j f(z, t-t_j); \tag{8}$$

where time delays $t_j$ are given by formula (4) and the corresponding probabilities $a_j$ by formula (7).

## 3.2. NOS Model. Even Rigidity

Of all models considered in this paper, this is the most complicated one in terms of the particle's dynamics. We will describe a typical trajectory of the particle for a given initial configuration of scatterers. Let $d_i$, $i = 1, 2, \ldots$ denote the positions of back-scatterers on $\mathbb{R}_+$ at time $t=0$. We assume $d_0 = 0$. Let $t_i$ be the times of the first visits of the particle to positions $d_i$, $i = 1, 2, \ldots$. As before, $\eta(z, t) = (s, i)$ shall denote the state of the scatterer at position $z$ at time $t$.

We will derive the first-visit equation for $z > 0$. The case $z < 0$ can be considered in a similar way. Let us suppose that at time $t$ the particle arrives at position $z$ for the first time.

If $\eta(z, t) = (0, 0)$, then after the particle's interaction with the scatterer the state of the scatterer will change to $\eta(z, t_+) = (0, 1)$. The particle will continue moving with velocity $v(t)$ and arrive at position $z+1$ at time $t+1$.

If $\eta(z, t) = (1, 0)$, i.e., there is a back-scatterer at $z$ then, according to the notations introduced earlier, $z = d_l$ and $t = t_l$ for some $l$, $1 \leqslant l \leqslant z$. The precise trajectory of the particle depends on several factors including the type of scatterer at the origin at time $t = 0$. We will describe the typical trajectory in the case $\eta(0, 0) = (0, 0)$. (The other case could be considered as well, however this factor does not affect the *character* of the particle's motion, nor does it change the final equation.) In the course of propagation from $z$ to $z+1$ the particle will perform a series of steps.

Let us denote as $y_k$ the position of the back-scatterer closest to $-k+1$ such that $y_k < -k+1$. Then during the first step the particle will bounce on the interval $[y_{d_{l-1}}, z]$ covering the length of $[0, z]$ $(r-1)$ times and the length of $[y_{d_{l-1}}, 0]$ $r$ times.

In step 2 it will bounce on the intervals $[0, 1]$ and $[-1, 2]$ covering the length of $[0, 1]$ $2r$ times and the length of $[-1, 2]$ $r$ times.

In each subsequent step $i$ the particle will bounce on intervals $[0, 1], [-1, 2],..., [-i, i+1]$, $i < z-1$ covering the length of $[0, 1]$ $2r$ times and the lengths of all other intervals $r$ times.

In $(z-1)$-st step, the behaviour of the particle will depend on the state of the scatterer at position $-z+1$. If $\eta(-z+1, t_l) = (1, 0)$ or $(1, r/2)$ then during this step it will bounce on $[0, 1],..., [-z+1, z]$ following the above pattern. If $\eta(-z+1, t_l) = (0, 0)$ then it will bounce on $[0, 1],..., [-z+2, z-1]$, $[y_z, z]$ instead, where $y_z$ is defined as above.

After this, the particle will return to $z$ and repeat this series of steps once again with minor or no changes to all but the last step. The sequence of the intervals involved in this last step will again depend on $\eta(-z+1, t_l)$. If $\eta(-z+1, t_l) = (1, 0)$ or $(0, 0)$ then the particle will bounce on intervals $[0, 1], [-1, 2],..., [-z+1, z]$, and if $\eta(-z+1, t_l) = (1, r/2)$ then it will bounce on $[0, 1],..., [-z+2, z-1]$, $[y_z, z]$. Upon finishing this step the particle will move to $z+1$.

To compute the delay times $t_s$, notice that during its propagation from $z$ to $z+1$ the particle covers the length of each interval $[-i+1, i]$ exactly $(2r)\, 2^{z-i}$ times for $i = 2,..., z$ and the length of $[y_z, -z+1]$ $r$ times. The length of $[0, 1]$ will be covered $2\sum_{i=2}^{z} (2r)\, 2^{z-i} = 4r(2^{z-1}-1)$ times. Let $d$ be the random variable corresponding to the distance between $-z+1$ and $y_z$. Then, for any $d = j$ the delay time $t_j$ can be computed as follows:

$$\begin{aligned} t_j &= \sum_{i=2}^{z} 2r2^{z-i}(2i-1) + 4r(2^{z-1}-1) + rj \\ &= r(j + 7 \cdot 2^z - 4z - 10), \qquad j \geqslant 0. \end{aligned} \tag{9}$$

It follows from our discussion above that time delay $t_j$, $j \geqslant 1$ occurs if the distance between $-z+1$ and $y_z$ is equal to $j$ and $\eta(-z+1, t) \neq (1, 0)$, whereas time delay $t_0$ occurs when $\eta(-z+1, t) = (1, 0)$. If we denote the probability of the event that $\eta(-z+1, t) = (1, 0)$ as $\mathscr{P}_{(1,0)}$ then

$$\begin{aligned} a_0 &= p\mathscr{P}_{(1,0)} \\ a_j &= p^2 q^{j-1}(1 - \mathscr{P}_{(1,0)}), \qquad j \geqslant 1 \end{aligned} \tag{10}$$

To compute these probabilitites we need to compute the value of $\mathscr{P}_{(1,0)}$. To do so we will first compute it for a given configuration of scatterers at time $t=0$ and then sum over all possible configurations with appropriate weights. Let $\mathscr{P}^{l}_{(1,0)}(d_1,..., d_l)$ denote the conditional probability of the event that $\eta(-z+1, t)=(1,0)$ given that at $t=0$ positions $d_i$, $1\leqslant i\leqslant l$ are occupied by back-scatterers, and $z=d_l$, $t=t_l$ for some $1\leqslant l\leqslant z$. The scatterer at position $-z+1$ will be in state $(1,0)$ at time $t$ in one of the two cases:

– if $\eta(-z+1,0)=(0,0)$ and the scatterer has been visited by the particle some time before $t_l$. Note that it could have been visited during any of the time intervals $[t_{i-1}, t_i)$, $2\leqslant i\leqslant l$ ($t_1=1$ for any configuration of scatterers, hence, the particle does not visit any sites outside $[0,1]$ during interval $[t_0, t_1)$).

– if $\eta(-z+1,0)=(1,0)$ and the scatterer has not been visited before time $t_l$;

The probability of the event that $(-z+1)$ was visited during time interval $[t_{i-1}, t_i)$ is equal to:

$$\begin{aligned}&\mathscr{P}\{\eta(k,0)=(0,0),\, -d_{i-1}\leqslant k\leqslant -d_l+2;\, \eta(-d_{i-1}+1, t_{i-1})\neq(1,0)\}\\&\quad=q^{d_l-d_{i-1}-1}(1-\mathscr{P}^{i-1}_{(1,0)}(d_1,..., d_{i-1})).\end{aligned}$$

Hence, the total probability $\mathscr{P}_v(d_1,..., d_l)$ of the event that the scatterer at $(-z+1)$ has been visited before time $t_l$ can be computed as a sum of the above probabilities over all time intervals $[t_{i-1}, t_i)$:

$$\mathscr{P}_v(d_1,..., d_l)=\sum_{i=2}^{l} q^{d_l-d_{i-1}-1}(1-\mathscr{P}^{i-1}_{(1,0)}(d_1,..., d_{i-1})).$$

Now, the probability $\mathscr{P}^{l}_{(1,0)}(d_1,..., d_l)$ can be computed as follows:

$$\begin{aligned}\mathscr{P}^{l}_{(1,0)}(d_1,..., d_l)&=q\mathscr{P}_v(d_1,..., d_l)+p(1-\mathscr{P}_v(d_1,..., d_l))\\&=p+(q-p)\sum_{i=2}^{l} q^{d_l-d_{i-1}-1}(1-\mathscr{P}^{i-1}_{(1,0)}(d_1,..., d_{i-1})),\\&\qquad 2\leqslant l\leqslant z,\\\mathscr{P}^{1}_{(1,0)}(d_1)&=p.\end{aligned}\tag{11}$$

This is a recursive equation on $\mathscr{P}^l_{(1,0)}(d_1,\ldots,d_l)$. To solve it, it is convenient to define a new quantity $B_i = q^{-d_i-1}(1-\mathscr{P}^i_{(1,0)}(d_1,\ldots,d_i))$. The equation for $B_l$ follows immediately from (11):

$$B_l = q^{-d_l} + \left(\frac{p}{q} - 1\right) \sum_{i=1}^{l-1} B_i,$$
$$B_1 = q^{-d_1}$$

and its solution is given by:

$$B_l = q^{-d_l} + \left(\frac{p}{q} - 1\right) \sum_{i=1}^{l-1} \left(\frac{p}{q}\right)^{l-i-1} q^{-d_i}.$$

Going back to $\mathscr{P}^l_{(1,0)}(d_1,\ldots,d_l)$ we get:

$$\mathscr{P}^l_{(1,0)}(d_1,\ldots,d_l) = \begin{cases} p, & l = 1, \\ p-(p-q)\sum_{i=2}^{l} \left(\frac{p}{q}\right)^{l-i} q^{d_l-d_{i-1}}, & 2 \leqslant l \leqslant z. \end{cases} \tag{12}$$

Finally, to compute $\mathscr{P}_{(1,0)}$ we will sum the probabilitites given by (12) over all possible configurations of scatterers, i.e.:

$$\begin{aligned} \mathscr{P}_{(1,0)} &= \sum_{l=1}^{z} p^l q^{z-l} \sum_{1 \leqslant d_1 < \cdots < d_{l-1} < d_l = z} \mathscr{P}^l_{(1,0)}(d_1,\ldots,d_l) \\ &= p^2 + \frac{q-p}{2p(1+q)}\,(p+2q^{2z+1}-(1+q)(p^2+q^2)^z). \end{aligned} \tag{13}$$

In the case $\eta(z,t)=(0,0)$ there is no time delay and the corresponding probability equals the probability of the event that $\eta(z,0)=(0,0)$. In order to conform to the notations used in Eq. (1), we will assign index $(-1)$ to the quantities corresponding to this case, hence:

$$\begin{aligned} t_{-1} &= 0, \\ a_{-1} &= q. \end{aligned} \tag{14}$$

It is easy to verify that $a_{-1} + \sum_{j \geqslant 0} a_j = 1$.

We are now ready to write the first visit equation for the NOS-model with even rigidity for $z>0$. Combining Eqs. (1), (10), and (14) we get:

$$\begin{aligned} f(z+1,t+1) &= qf(z,t) + p\mathscr{P}_{(1,0)}\, f(z,t-t_0) \\ &\quad + p^2(1-\mathscr{P}_{(1,0)}) \sum_{j=1}^{\infty} q^{j-1} f(z,t-t_j), \end{aligned} \tag{15}$$

where probability $\mathscr{P}_{(1,0)}$ is given by (13) and time delays $t_j$ by (9). A similar equation can be obtained for $z<0$.

### 3.3. OS Model

We begin with the description of a typical trajectory of the particle in the OS-model. We will use the same notation $\eta(z,t)=(s,i)$ for the state of the scatterer occupying position $z$, at time $t$. Here $s\in\{0,1\}$ (now 0 corresponds to a right rotator, and 1 to a left rotator) and $i$ denotes the index of the scatterer. Let $L_z$ be the random variable corresponding to the number of left rotators on the interval $[1,z]$ at time $t=0$, if $z>0$, and $R_z$ be the random variable corresponding to the number of right rotators on the interval $[z,0]$, if $z\leqslant 0$.

First we consider the case when $z>0$. Let us suppose now that at time $t$ the particle arrives at position $z$ for the first time. There are two possible outcomes.

If $\eta(z,t)=(0,0)$, then after the particle's interaction with the scatterer the state of the scatterer will change to $\eta(z,t_+)=(0,1)$. The particle will continue moving with velocity $v(t)$ and arrive at position $z+1$ at time $t+1$.

If $\eta(z,t)=(1,0)$, then the particle will be reflected by the scatterer at $z$. Due to the presence of the right rotator at $z-1$, the particle will bounce between positions $z$ and $z-1$ until this rotator flips. By this time, the particle will have covered the distance between these two positions $(2r-1)$ times, and will be located at $z-1$. Thus, it will be sent further to the left. A similar process will take place on the intervals $[z-2,z-1]$,..., $[1,2]$. After returning to position 1, it will be scattered to the left once again, where it will travel to the nearest position $y\leqslant 0$ with $R_y=L_z$ and $\eta(y,t)=(0,0)$. Since $y$ is occupied by the right rotator the particle will be reflected to the right. Next, it will bounce on each of the intervals $[y,y+1]$, $[y+1,y+2]$,..., $[0,1]$ in the same way it did on the positive semiaxis. After returning to position 1 it will travel directly to position $z+1$. If $T_{z,z+1}$ is the time that takes the particle to propagate from $z$ to $z+1$ then the configuration of scatterers after $T_{z,z+1}$ time steps is given by:

$$\eta(x,t+T_{z,z+1})=(1,0), \qquad y\leqslant x\leqslant 0,$$
$$\eta(x,t+T_{z,z+1})=(0,1), \qquad 1\leqslant x\leqslant z.$$

Note that in the course of propagation the particle covers the length of each interval $[i,i+1]$, $y\leqslant i\leqslant z-1$, $2r$ times.

Let us compute the time delays $t_s$ in Eq. (1) and their probabilities $a_s$, starting with the case $\eta(z,t)=(1,0)$. For each $L_z=l$, $1\leqslant l\leqslant z$, let $d_l$ be the random variable that corresponds to the distance between position $y \leqslant 0$, such that $R_y = l$ and $\eta(y, 0) = (0, 0)$, and the origin. Note that $d_l \geqslant l-1$. It follows from the observations above that the time required for the particle to propagate from $z$ to $z+1$, for each $d_l = j$ is equal to

$$t_{lj} = 2r(z+j), \qquad j \geqslant l-1. \tag{16}$$

The delay $t_{lj}$ occurs if at time $t=0$ there were exactly $l-1$ right scatterers on the interval $[-(j-1), 0]$ and $\eta(-j, 0) = (0, 0)$. There are $C_j^{l-1}$ such configurations for each $l$. The probability of the event that $L_z = l$ is equal to the probability that there were exactly $l-1$ left scatterers on the interval $[1, z-1]$ at $t=0$ and $\eta(z, 0) = (1, 0)$. If $q$ is the probability that a site is occupied by a right rotator at time $t=0$ and $p = 1-q$, then the probability of delay $t_{lj}$ is given by

$$a_{lj} = (C_{z-1}^{l-1} p^l q^{z-l})(C_j^{l-1} q^l p^{j+1-l}), \qquad 1 \leqslant l \leqslant z,\ j \geqslant l-1. \tag{17}$$

In order to be consistent with the notations used in equation (1) we assign index $s=0$ to the quantities associated with the case when $\eta(z, t) = (0, 0)$. Then

$$t_0 = 0, \qquad \text{and} \qquad a_0 = q. \tag{18}$$

It can be verified that $a_0 + \sum_{l=1}^{z} \sum_{j \geqslant l-1} a_{lj} = 1$.

Now, we can combine Eqs. (1) and (16)–(18) to write the first-visit equation for the OS-model for $z > 0$:

$$f(z+1, t+1) = q f(z, t) + \sum_{l=1}^{z} \sum_{j=l-1}^{\infty} C_{z-1}^{l-1} C_j^{l-1} q^z p^{j+1} f(z, t-2r(z+j)). \tag{19}$$

Changing the order of summation and computing the internal sum in the resulting equation reduces (19) to:

$$f(z+1, t+1) = q f(z, t) + \left(\frac{q}{p}\right)^z \sum_{k=z}^{\infty} C_{k-1}^{z-1} p^{k+1} f(z, t-2rk), \tag{20}$$

where the new index of summation $k$ is related to the old index $j$ via the relation: $k = j+z$.

The equation for $z \leqslant 0$ can be obtained in the same way as it was done for positive $z$ and reads:

$$f(z-1, t+1) = p f(z, t) + \left(\frac{p}{q}\right)^{|z|+2} \sum_{k=|z|+2}^{\infty} C_{k-1}^{|z|+1} q^{k+1} f(z, t-2rk). \tag{21}$$

## 4. CONTINUOUS LIMITS

Let us consider Eq. (2), which is a slightly more general version of Eq. (1) where the distance between neighboring sites on $\mathbb{R}$ is $\zeta$ and the elementary time step is $\tau$. The distribution function of time delays in the model described by this equation, allows us to define the following two quantities:[11] the average displacement time (the average time taken by the particle to perform a diplacement from $z$ to $z+\zeta$)

$$\langle t_D \rangle = \sum_s a_s(t_s+1)\,\tau = \tau(\langle t_s \rangle + 1),$$

and the variance

$$\mathrm{Var}(t_D) = \langle (t_s+1)^2 \rangle\,\tau^2 - \langle t_D \rangle^2 = \tau^2(\langle t_s^2 \rangle - \langle t_s \rangle^2).$$

In order to find the continuous limit of the finite-difference equation (1) we can expand Eq. (2) in powers of $\zeta$ and $\tau$ and then pass to the limit $\zeta, \tau \to 0$. The argument similar to the one used in ref. 11 shows that the continuous limit of Eq. (1) obtained in this manner has the following form:

$$\partial_z f(z,t) + \frac{1}{c}\,\partial_t f(z,t) = \frac{\gamma}{2}\,\partial_t^2 f(z,t), \tag{22}$$

where

$$\frac{1}{c} = \lim_{\zeta \to 0,\, \tau \to 0} \frac{\langle t_D \rangle}{\zeta}, \qquad \gamma = \lim_{\zeta \to 0,\, \tau \to 0} \frac{\mathrm{Var}(t_D)}{\zeta}, \tag{23}$$

and the limit $\zeta, \tau \to 0$ is subject to the condition that both $\gamma$ and $1/c$ be finite. Note that the form of Eq. (22) does not depend on whether the summation in (1) is performed over a finite or an infinite number of configurations.

### 4.1. NOS Model. Odd Rigidity

The time delays $t_j$ and their probabilities $a_j$ in the model with $\zeta = 1$ and $\tau = 1$ have been computed in the previous section. In order to obtain the corresponding quantities for the case of arbitrary $\zeta$ and $\tau$ let us make two observations. First, the quantities $t_j$ computed earlier represent the number of *time steps* that it takes the particle to flip the scatterer at position $z$ and start moving to the next position on the lattice. Thus, the values of $t_j$, $j \geqslant 1$ do not depend on the value of the elementary time step $\tau$ and, therefore, will be given by formulae (4) and (6) for any $\tau$. Second, for $r=1$ the probabilities $a_j$ given by (3) do not depend on the distance between neighboring sites. Hence they will also stay the same after the rescaling. For the case $r>1$, however, we need to replace $z$ with $N=z/\zeta$ in the formula for $a_j$, as $N$ represents the number of lattice unit lengths between the site with coordinate $z$ and the origin for an arbitrary $\zeta$. The values of $a_j$ for this case are given by

$$a_j = \begin{cases} q, & j=1; \\ p^2 q^{j-2}, & 2 \leqslant j \leqslant N-1; \\ p^2 q^{j-1} + p^2 q^{j-1} = 2p^2 q^{j-1}, & j=N; \\ p^2 q^{j-1} + p^4 q^{j-3} = p^2(p^2+q^2)\, q^{j-3}, & j \geqslant N+1. \end{cases}$$

Note that in the limit $\zeta \to 0$, $N$ becomes unbounded.

The computations of $\langle t_D \rangle$ and $\operatorname{Var}(t_D)$ are fairly straighforward and shall be omitted here for the sake of brevity. The results are presented below:

$$\langle t_D \rangle = \tau(r(1+2p) + \mathcal{O}(Nq^N)),$$

$$\operatorname{Var}(t_D) = \tau^2 \frac{q}{p}((r(1+2p)-1)^2 + (r-1)^2 + \mathcal{O}(Nq^N)), \qquad \text{as} \quad N \to \infty.$$

These formulae are valid for any odd $r \geqslant 1$.

This allows us to compute $c$ and $\gamma$ in Eq. (22). Substituting the values obtained for $\langle t_D \rangle$ and $\operatorname{Var}(t_D)$ into (23) and taking the limit $\zeta \to 0$, $\tau \to 0$ yields:

$$\frac{1}{c} = r(1+2p) \lim_{\zeta \to 0,\, \tau \to 0} \frac{\tau}{\zeta},$$

$$\gamma = \frac{q}{p}((r(1+2p)-1)^2 + (r-1)^2) \lim_{\zeta \to 0,\, \tau \to 0} \frac{\tau^2}{\zeta}.$$

The condition of boundedness of these two quantities requires that $\lim_{\zeta \to 0,\, \tau \to 0}(\tau/\zeta)$ be finite. We choose

$$\lim_{\zeta \to 0,\, \tau \to 0} \frac{\tau}{\zeta} = 1$$

to preserve the scaling factor of 1 obtained in the case when $\zeta = \tau = 1$. Clearly $\gamma = 0$ under this condition, so the continuous limit of Eq. (8) reads:

$$\partial_z f(z,t) + r(1+2p)\, \partial_t f(z,t) = 0. \tag{24}$$

This equation describes propagation on the real line with speed $(r(1+2p))^{-1}$. Since the expected value of the particle's position is proportional to time its second moment grows quadratically. Therefore, in the continuous limit, the NOS-model with odd rigidity exhibits a super-diffusive behaviour.

## 4.2. NOS Model. Even Rigidity

The time delays $t_j$ and probabilitites $a_j$ with which they occur for the case of an arbitrary $\zeta$ and $\tau$ can be obtained from (9)–(10) by substituting $z$ in those formulae with $N = z/\zeta$. Hence, for $z > 0$ we have

$$t_j = \begin{cases} 0, & j = -1; \\ r(j+7\cdot 2^N - 4N - 10), & j \geqslant 0, \end{cases}$$

$$a_j = \begin{cases} q, & j = -1 \\ p\mathscr{P}_{(1,0)}, & j = 0 \\ p^2 q^{j-1}(1-\mathscr{P}_{(1,0)}), & j \geqslant 1 \end{cases}$$

where

$$\mathscr{P}_{(1,0)} = p^2 + \frac{q-p}{2p(1+q)}(p + 2q^{2N+1} - (1+q)(p^2+q^2)^N).$$

Using these formulae we can compute the average displacement time and its variance:

$$\langle t_D \rangle = \tau(7rp\,2^N + \mathscr{O}(N)),$$
$$\operatorname{Var}(t_D) = \tau^2(49r^2pq\,2^{2N} + \mathscr{O}(N2^N)).$$

Dividing these by $\zeta$ and taking the limit $\zeta \to 0$, $\tau \to 0$ we find the coefficients in Eq. (22):

$$\frac{1}{c} = 7rp \lim_{\zeta\to 0,\, \tau\to 0} \frac{2^{z/\zeta}\tau}{\zeta},$$

$$\gamma = 49r^2p\,q \lim_{\zeta\to 0,\, \tau\to 0} \frac{2^{2z/\zeta}\tau^2}{\zeta}.$$

The condition of boundedness of these quantities requires that both limits in the above expressions be finite. Since $z$ enters the above limits exponentially we will choose

$$\lim_{\zeta\to 0,\, \tau\to 0} \frac{2^{z/\zeta}\tau}{\zeta} = 2^z$$

or, equivalently, $\tau = \mathcal{O}(\zeta\, 2^{-1/\zeta})$. Under this condition $\gamma = 0$, $1/c = 7rp2^z$ and the continuous limit of Eq. (15) for $z > 0$ reads:

$$f_z(z, t) + 7rp\, 2^z f_t(z, t) = 0. \tag{25}$$

Similarly, one can show that the continuous limit of the kinetic equation corresponding to the case $z \leqslant 0$ is given by

$$f_z(z, t) - 7rp\, 2^{|z|} f_t(z, t) = 0. \tag{26}$$

This equation can also be obtained by changing the sign of $z$ in Eq. (25) to the opposite which reflects the symmetry of the particle's motion with respect to the origin.

Equations (25) and (26) can be combined into a single equation valid for all $z \in \mathbb{Z}$ as follows:

$$f_z(z, t) + \mathrm{sgn}(z)\, 7rp\, 2^{|z|} f_t(z, t) = 0. \tag{27}$$

It can be shown that the second moment of the particle's position in the process described by (27) grows logarithmically with time. Hence, the continuous limit of the NOS-model with even rigidity exhibits a subdiffusive behaviour which is somewhat similar to that of the Ornstein–Uhlenbeck process.

### 4.3. OS Model

The delay times $t_{lj}$ and their probabilities $a_{lj}$ for the case when the elementary lattice length is $\zeta$ and elementary time step is $\tau$ can again be obtained by replacing $z$ with $N = z/\zeta$ in the expressions derived in the previous section. Hence, for $z > 0$ we have:

$$\begin{cases} t_0 = 0, \\ t_{lj} = 2r(N + j), \qquad 1 \leqslant l \leqslant z,\ j \geqslant l - 1, \end{cases}$$

and

$$\begin{cases} a_0 = q, \\ a_{lj} = C_{N-1}^{l-1} C_j^{l-1} q^N p^{j+1} \qquad 1 \leqslant l \leqslant z,\ j \geqslant l - 1. \end{cases}$$

Note that, as before, in the limit $\zeta \to 0$, $N$ becomes unbounded.

By straightforward computation we obtain the following formulae for $\langle t_D \rangle$ and $\mathrm{Var}(t_D)$:

$$\langle t_D \rangle = \tau \left( 2r \frac{p}{q} N + 1 \right),$$

$$\mathrm{Var}(t_D) = 4\tau^2 \left( \frac{p}{q} N^2 + N \left( \frac{p}{q} \right)^2 \right).$$

Now, dividing $\langle t_D \rangle$ and $\mathrm{Var}(t_D)$ above by $\zeta$ and taking the limit $\zeta \to 0$, $\tau \to 0$ we get:

$$\frac{1}{c} = 2r \frac{p}{q} z \lim_{\zeta \to 0,\, \tau \to 0} \frac{\tau}{\zeta^2},$$

$$\gamma = 4r^2 \frac{p}{q} z^2 \lim_{\zeta \to 0,\, \tau \to 0} \frac{\tau^2}{\zeta^3}.$$

For these quantities to stay bounded, both of the limits above must be finite. Hence we will require that

$$\lim_{\zeta \to 0,\, \tau \to 0} \frac{\tau}{\zeta^2} = 1. \tag{28}$$

Under this condition $\gamma = 0$ and the expression for $1/c$ simplifies to:

$$\frac{1}{c} = 2r \frac{p}{q} z. \tag{29}$$

Substituting $1/c$ given by (29) and $\gamma = 0$ into (22) we obtain the continuous limit of Eq. (20):

$$f_z(z, t) + 2r \frac{p}{q} z f_t(z, t) = 0. \tag{30}$$

Similarly, the continuous limit of Eq. (21) reads

$$f_z(z, t) - 2r \frac{q}{p} z f_t(z, t) = 0. \tag{31}$$

The above two equations can be combined into a single equation valid for all $z \in \mathbb{Z}$:

$$f_z(z, t) + 2r \left( \frac{p}{q} \right)^{\mathrm{sgn}(z)} |z| \, f_t(z, t) = 0. \tag{32}$$

In the model described by this equation the second moment of the particle's position grows linearly with time. Hence in the continuous limit the OS-model exhibits a diffusive behaviour.

## 5. CONCLUDING REMARKS

As we mentioned in the Introduction, deterministic walks in random environments are generated by two processes: the deterministic motion in the region $D_t$ formed by all vertices visited by the particle to the moment $t$, and a random process of propagation into the complement $\bar{D}_t$ of this region. Note that in general this region changes with time. Equation (1) is a general equation that describes the motion of the boundary of this deterministic region. The models considered in this paper demonstrate sub-diffusive, diffusive and super-diffusive types of behaviour for the boundary that consists of only two points.

It has been stated in ref. 11 that propagation occurs only if the sum in Eq. (1) has a finite number of terms, and that this condition guarantees the existence of a blocking mechanism responsible for the propagation. However, the phenomenon of propagation occurs as well if the sum in (1) contains an infinite number of terms. In fact, the NOS-model with an odd rigidity[1] provides a counterexample to that statement. This model also demonstrates that the blocking mechanism can exist when the sum in (1) contains an infinite number of terms.

Note that Eq. (1) is valid for any dimension. Indeed, we can consider an infinite ordered set of vertices $V$ such that $V \cap D_t$ has only one connected component for any $t \geqslant 0$, i.e., the boundary $\partial(V \cap D_t) \in \partial D_t$ consists of two points. Then the validity of (1) for propagation on $\mathbb{Z}$ follows trivially.

The deterministic character of motion inside region $D_t$ is the reason why continuous limits of the kinetic equations for all of the models explored in this paper contain only the propagation term. Indeed, the motion of the particle inside the deterministic region $D_t$ has no fluctuations and the fluctuations on the boundary $\partial D_t$ generated by random distribution of scatterers outside of $D_t$ are relatively small. This situation should be compared with the biased random walk (see, e.g., ref. 9) whose continuous limit contains a fluctuation term due to the pure randomness of the corresponding model. Random walks allow for an appropriate scaling of probabilities of transitions[9] while no such scaling exists for deterministic walks in random environments.

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